To appear in *J. Funct. Anal.*

Spectral asymptotics of the Laplacian on supercritical bond-percolation graphs ⋆

Peter Müller [a] and Peter Stollmann [b]

[a] *Institut für Theoretische Physik, Georg-August-Universität, 37077 Göttingen, Germany*
[b] *Technische Universität Chemnitz, Fakultät für Mathematik, 09107 Chemnitz, Germany*

Version of 6 February 2008

Abstract

We investigate Laplacians on supercritical bond-percolation graphs with different boundary conditions at cluster borders. The integrated density of states of the Dirichlet Laplacian is found to exhibit a Lifshits tail at the lower spectral edge, while that of the Neumann Laplacian shows a van Hove asymptotics, which results from the percolating cluster. At the upper spectral edge, the behaviour is reversed.

Key words: Laplacian, Percolation, Integrated density of states

1 Introduction and summary

Ever since Mark Kac posed the question "Can one hear the shape of a drum?" [15], there has been a great deal of interest in finding relations between the geometry of a manifold or a graph and spectral properties of the Laplacian defined on it. The impressive works [8–10,6,3], which have been chosen by way of example, witness the steady progress achieved in recent years and provide further references. Whereas Laplacians on manifolds dominated the scene in the earlier years, the rise of spectral graph theory [23,22,11,5,7] in the late 1980s and 90s has contributed to deepen our understanding of the discrete case.

⋆ Work supported by the Deutsche Forschungsgemeinschaft under grant nos. Mu 1056/2-1 and Sto 294/3-1.

Email addresses: `peter.mueller@physik.uni-goe.de` (Peter Müller), `p.stollmann@mathematik.tu-chemnitz.de` (Peter Stollmann).

Spectral theory of random graphs, however, is still a widely open field. The very recent contributions [21,2,14] take a probabilistic point of view to derive heat-kernel estimates for Laplacians on *supercritical* Bernoulli bond-percolation graphs in the d-dimensional hyper-cubic lattice. On the other hand, traditional methods from spectral theory are used in [19] to investigate the integrated density of states of Laplacians on *subcritical* bond-percolation graphs. Depending on the boundary condition that is chosen at cluster borders, two different types of *Lifshits asymptotics* at spectral edges were found [19]. For example, the integrated density of states of the Neumann Laplacian behaves as

$$\text{“ } N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0) \sim \exp\{-E^{-1/2}\} \text{ ”} \qquad \text{as } \; E \downarrow 0 \tag{1.1}$$

at the lower spectral edge for bond probabilities p below the percolation threshold p_c. We have put quotation marks here, because, strictly speaking, one should take appropriate logarithms on both sides. The Lifshits exponent $1/2$ in (1.1) is independent of the spatial dimension d. This was explained by the fact that, asymptotically, N_{N} is dominated by the smallest eigenvalues which arise from very long *linear clusters* in this case. In contrast, for the Dirichlet Laplacian and $p < p_c$, it was found that

$$\text{“ } N_{\mathrm{D}}(E) \sim \exp\{-E^{-d/2}\} \text{ ”} \qquad \text{as } \; E \downarrow 0\,. \tag{1.2}$$

We note that $N_{\mathrm{D}}(0) = 0$. The Lifshits exponent in (1.2) comes out as $d/2$, because the dominating small Dirichlet eigenvalues arise from large *fully connected cube- or sphere-like clusters*. Thus, depending on the boundary condition (and the spectral edge) different geometric graph properties show up in the integrated density of states. We refer to the literature cited in [19] for a discussion of other spectral properties of these and closely related operators, for the history of the problem and what is known in the physics literature. Lifshits asymptotics for a Neumann Laplacian on Erdős–Rényi random graphs are studied in [18].

In this paper we pursue the investigations of [19] and ask what happens to (1.1) and (1.2) in the *supercritical phase* of bond-percolation graphs. Clearly, one would not expect the contribution of the finite clusters to alter the picture completely. But for the infinite percolating cluster, the story may be different. Indeed, we will prove that the percolating cluster produces a *van Hove asymptotics*

$$\text{“ } N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0) \sim E^{d/2} \text{ ”} \qquad \text{as } \; E \downarrow 0 \tag{1.3}$$

in the Neumann case for $p > p_c$. There is also an additional Lifshits-tail behaviour due to finite clusters, but it is hidden under the dominating asymptotics (1.3). Loosely speaking, (1.3) is true because the percolating cluster looks like the full regular lattice on very large length scales (bigger than the correlation length) for $p > p_c$. On smaller scales its structure is more like that

of a jagged fractal. The Neumann Laplacian does not care about these small-scale holes, however. All that is needed for (1.3) to be true is the existence of a suitable d-dimensional, infinite grid. In contrast, the Dirichlet Laplacian does care about holes at all scales so that (1.2) continues to hold for $p \geqslant p_c$, as we shall prove. Low-lying Dirichlet eigenvalues require large fully connected cube- or sphere-like regions, and this is a large-deviation event.

Closely related large-deviation results for Laplacians on percolation graphs have been obtained in [1,4]. To be precise, [1,4] refer to the Pseudo-Dirichlet Laplacian $\Delta_{\widetilde{\mathrm{D}}}$ in the sense of our Definition 2.1(ii) below. Considering both site- and bond-percolation graphs, and using a discrete version of the method of enlargement of obstacles, Antal [1] derives the long-time asymptotics for the mean (i.e. annealed) hitting-time distribution of the set of absent sites (resp. bonds) for the random walk generated by $\Delta_{\widetilde{\mathrm{D}}}$. Biskup and König work in the setting of the parabolic Anderson model, which contains $\Delta_{\widetilde{\mathrm{D}}}$ on *site*-percolation graphs as a special case. In particular, they establish a Lifshits tail for the corresponding integrated density of states, see also Remark 2.6 (v).

This paper is organised as follows. In the next section we give a precise statement of our results in Theorems 2.5 and 2.7. Section 3 is devoted to the proof of Theorem 2.5. In this proof we follow the strategy laid down in [24], see also [25]. The goal there was to establish Lifshits tails in the context of random Schrödinger operators. Finally, Section 4 contains the proof of Theorem 2.7, where we apply the recent deep heat-kernel estimates from [21,2,14].

2 Definitions and precise formulations

To set up the mathematical arena, let us first recall some notions from Bernoulli bond percolation. For $d \in \mathbb{N}$, a natural number, we denote by $\mathbb{L}^d$ the (simple hyper-cubic) lattice in d dimensions. Being a graph, the lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ has the *vertex set* $\mathbb{Z}^d$ and the *edge set* $\mathbb{E}^d$ given by all unordered pairs $\{x, y\}$ of nearest-neighbour vertices $x, y \in \mathbb{Z}^d$, that is, those vertices which have Euclidean distance $|x - y| := \Big(\sum_{\nu=1}^{d} |x_\nu - y_\nu|^2\Big)^{1/2} = 1$. Here, elements of $\mathbb{Z}^d$ are canonically represented as d-tuples $x = (x_1, \ldots, x_d)$ with entries from $\mathbb{Z}$. Next, we consider the probability space $\Omega = \{0,1\}^{\mathbb{E}^d}$, which is endowed with the usual product sigma-algebra, generated by finite cylinder sets, and equipped with a product probability measure $\mathbb{P}$. Elementary events in Ω are sequences of the form $\omega \equiv (\omega_{\{x,y\}})_{\{x,y\} \in \mathbb{E}^d}$, and we assume their entries to be independently and identically distributed according to a Bernoulli law $\mathbb{P}(\omega_{\{x,y\}} = 1) = p$ with *bond probability* $p \in]0, 1[$. To a given $\omega \in \Omega$, we associate an edge set $\mathscr{E}^{(\omega)} := \Big\{\{x, y\} \in \mathbb{E}^d : \omega_{\{x,y\}} = 1\Big\}$.

A *bond-percolation graph in* $\mathbb{Z}^d$ is the mapping $\mathscr{G} : \Omega \ni \omega \mapsto \mathscr{G}^{(\omega)} := (\mathbb{Z}^d, \mathscr{E}^{(\omega)})$ with values in the set of subgraphs of $\mathbb{L}^d$. Given $x \in \mathbb{Z}^d$, the *vertex degree* $d_{\mathscr{G}^{(\omega)}}(x)$ counts the number of edges in $\mathscr{G}^{(\omega)}$ which share x as a common vertex.

Definition 2.1. The random operators $D : \Omega \ni \omega \mapsto D^{(\omega)}$, respectively $A : \Omega \ni \omega \mapsto A^{(\omega)}$, are called vertex-degree operator, respectively adjacency operator, of bond-percolation graphs in $\mathbb{Z}^d$. Their realisations, $D^{(\omega)} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, respectively $A^{(\omega)} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$, act on the Hilbert space of complex-valued, square-summable sequences indexed by $\mathbb{Z}^d$ according to

$$\begin{aligned} D^{(\omega)}\varphi(x) &:= d_{\mathscr{G}^{(\omega)}}(x)\,\varphi(x)\,, \\ A^{(\omega)}\varphi(x) &:= \sum_{y\in\mathbb{Z}^d:\ \{x,y\}\in\mathscr{E}^{(\omega)}} \varphi(y)\,, \end{aligned} \tag{2.1}$$

for all $\varphi \in \ell^2(\mathbb{Z}^d)$, all $x \in \mathbb{Z}^d$ and all $\omega \in \Omega$. With these definitions, we introduce *Laplacians on bond-percolation graphs* for three different "boundary conditions" at non-fully connected vertices

(i) *Neumann Laplacian*: $\Delta_{\mathrm{N}} := D - A$,

(ii) *Pseudo-Dirichlet Laplacian*: $\Delta_{\widetilde{\mathrm{D}}} := \Delta_{\mathrm{N}} + (2d\mathbb{1} - D) = 2d\mathbb{1} - A$,

(iii) *Dirichlet Laplacian*: $\Delta_{\mathrm{D}} := \Delta_{\mathrm{N}} + 2(2d\mathbb{1} - D)$.

Here $\mathbb{1}$ stands for the identity operator on $\ell^2(\mathbb{Z}^d)$.

Remarks 2.2. (i) The motivation and origin of the terminology for the different boundary conditions are discussed in [19] – together with some basic properties of the operators.

(ii) The random self-adjoint Laplacians are ergodic with respect to $\mathbb{Z}^d$-translations. Hence, their spectra and the spectral subsets arising in the Lebesgue decomposition are all equal to non-random sets with probability one. In particular, the spectrum is $\mathbb{P}$-almost surely given by $\mathrm{spec}(\Delta_{\mathrm{X}}) = [0, 4d]$ for all $\mathrm{X} \in \{\mathrm{N}, \widetilde{\mathrm{D}}, \mathrm{D}\}$, as was shown in [19].

Next, we define the quantity of our main interest for this paper, the integrated density of states of Δ_{X}. To this end let $\delta_x \in \ell^2(\mathbb{Z}^d)$ be the sequence which is concentrated at the point $x \in \mathbb{Z}^d$, i.e. $\delta_x(x) := 1$ and $\delta_x(y) := 0$ for all $y \in \mathbb{Z}^d \setminus \{x\}$. Moreover, Θ stands for the Heaviside unit-step function, which we choose to be right continuous, viz. $\Theta(E) := 0$ for all real $E < 0$ and $\Theta(E) := 1$ for all real $E \geqslant 0$.

Definition 2.3. For every $p \in]0,1[$ and every $\mathrm{X} \in \{\mathrm{N}, \widetilde{\mathrm{D}}, \mathrm{D}\}$ we call the

function

$$N_{\mathrm{X}} : \mathbb{R} \ni E \mapsto N_{\mathrm{X}}(E) := \int_{\Omega} \mathbb{P}(\mathrm{d}\omega)\ \langle \delta_0, \Theta\Big(E - \Delta_{\mathrm{X}}^{(\omega)}\Big)\delta_0\rangle \tag{2.2}$$

with values in the interval $[0,1]$ the *integrated density of states of* Δ_{X}.

Remarks 2.4. (i) The integrated density of states N_{X} is the right-continuous distribution function of a probability measure on $\mathbb{R}$. The set of its growth points coincides with the $\mathbb{P}$-almost-sure spectrum $[0, 4d]$ of Δ_{X}.

(ii) It is shown in [19] that the Laplacians are related to each other by a unitary involution, which implies the symmetries

$$\begin{aligned} N_{\widetilde{\mathrm{D}}}(E) &= 1 - \lim_{\varepsilon \uparrow 4d-E} N_{\widetilde{\mathrm{D}}}(\varepsilon)\,, \\ N_{\mathrm{D(N)}}(E) &= 1 - \lim_{\varepsilon \uparrow 4d-E} N_{\mathrm{N(D)}}(\varepsilon) \end{aligned} \tag{2.3}$$

for their integrated densities of states for all $E \in \mathbb{R}$. The limits on the right-hand sides of (2.3) ensure that the discontinuity points of N_{X} are approached from the correct side.

(iii) By ergodicity, Definition 2.3 of the integrated density of states coincides with the usual one in terms of a macroscopic limit of a finite-volume eigenvalue counting function. More precisely, let $\Lambda \subset \mathbb{Z}^d$ stand for bounded cubes centred at the origin with volume $|\Lambda|$. For every $\mathrm{X} \in \{\mathrm{N}, \widetilde{\mathrm{D}}, \mathrm{D}\}$ let $\Delta_{\mathrm{X},\Lambda}$ be the finite-volume restriction of Δ_{X} to $\ell^2(\Lambda)$ introduced in Def. 1.11 in [19]. Then there exists a set $\Omega' \subset \Omega$ of full probability, $\mathbb{P}(\Omega') = 1$, such that

$$N_{\mathrm{X}}(E) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left[\frac{1}{|\Lambda|}\ \mathrm{trace}_{\ell^2(\Lambda)}\, \Theta\Big(E - \Delta_{\mathrm{X},\Lambda}^{(\omega)}\Big) \right] \tag{2.4}$$

holds for all $\omega \in \Omega'$ and all $E \in \mathbb{R}$, except for the (at most countably many) discontinuity points of N_{X}, see Lemma 1.12 in [19]. In Section 3 we will construct another finite-volume restriction of $\Delta_{\widetilde{\mathrm{D}}}$, for which (2.4) holds, too.

Let $p_c \equiv p_c(d)$ denote the critical bond probability of the percolation transition in $\mathbb{Z}^d$. We recall that $p_c = 1$ for $d = 1$, otherwise $p_c \in]0, 1[$, see e.g. [13]. Despite the title of this paper, our first main result covers the non-percolating phase $p \in]0, p_c[$ and the critical point $p = p_c$, too.

Theorem 2.5. *Assume $d \in \mathbb{N}$ and $p \in]0, 1[$. Then the integrated density of states N_{X} of the Laplacian Δ_{X} on bond-percolation graphs in $\mathbb{Z}^d$ exhibits a* Lifshits tail *at the lower spectral edge*

$$\lim_{E \downarrow 0} \frac{\ln |\ln N_{\mathrm{X}}(E)|}{\ln E} = -\,\frac{d}{2} \qquad \textit{for} \quad \mathrm{X} \in \{\widetilde{\mathrm{D}}, \mathrm{D}\} \tag{2.5}$$

and at the upper spectral edge

$$\lim_{E \uparrow 4d} \frac{\ln|\ln[1 - N_{\mathrm{X}}(E)]|}{\ln(4d - E)} = -\frac{d}{2} \qquad \textit{for} \quad \mathrm{X} \in \{\mathrm{N}, \widetilde{\mathrm{D}}\}\,. \tag{2.6}$$

Remarks 2.6. (i) The theorem follows directly from the upper and lower bounds in Lemma 3.1 below, together with the subsequent Remark 2.6 (ii). In fact, the bounds of Lemma 3.1 provide a slightly stronger statement than Theorem 2.5.

(ii) The Lifshits tails at the upper spectral edge are related to the ones at the lower spectral edge by the symmetries (2.3).

(iii) In the non-percolating phase, $p \in]0, p_c[$, the content of the theorem is known from [19], where it is proved by a different method. The method of [19], however, does not seem to extend to the critical point or the percolating phase, $p \in]p_c, 1[$.

(iv) The Lifshits asymptotics of Theorem 2.5 are determined by those parts of the percolation graphs, which contain large, fully-connected cubes. This also explains why the spatial dimension enters the Lifshits exponent $d/2$.

(v) We expect that (2.5) can be refined in the case $\mathrm{X} = \widetilde{\mathrm{D}}$ as to obtain the constant

$$\lim_{E \downarrow 0} \frac{\ln N_{\widetilde{\mathrm{D}}}(E)}{E^{-d/2}} =: -c_*(d, p)\,. \tag{2.7}$$

An analogous statement is known from Thm. 1.3 in [4] for the case of *site*-percolation graphs. Moreover, it is demonstrated in [1] that the bond- and the site-percolation cases have similar large-deviation properties.

Our second main result complements Theorem 2.5 in the percolating phase.

Theorem 2.7. *Assume $d \in \mathbb{N} \setminus \{1\}$ and $p \in]p_c, 1[$. Then the integrated density of states of the Neumann Laplacian Δ_{N} on bond-percolation graphs in $\mathbb{Z}^d$ exhibits a* van Hove asymptotics *at the lower spectral edge*

$$\lim_{E \downarrow 0} \frac{\ln[N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0)]}{\ln E} = \frac{d}{2}\,, \tag{2.8}$$

while that of the Dirichlet Laplacian Δ_{D} exhibits one at the upper spectral edge

$$\lim_{E \uparrow 4d} \frac{\ln[N_{\mathrm{D}}^{-}(4d) - N_{\mathrm{D}}(E)]}{\ln(4d - E)} = \frac{d}{2}\,, \tag{2.9}$$

where $N_{\mathrm{D}}^{-}(4d) := \lim_{E \uparrow 4d} N_{\mathrm{D}}(E) = 1 - N_{\mathrm{N}}(0)$.

Remarks 2.8. (i) The theorem follows directly from the upper and lower bounds in Lemma 4.1 below, together with the symmetries (2.3). In fact, the bounds of Lemma 4.1 provide a slightly stronger statement than

Theorem 2.7. Lemma 4.1 relies mainly on recent estimates [21,2,14] for the long-time decay of the heat kernel of Δ_{N} on the percolating cluster.

(ii) The reference value $N_{\mathrm{N}}(0)$ in (2.8) results from the mean number density of zero eigenvalues of the Neumann Laplacian [19]. It is given by

$$N_{\mathrm{N}}(0) = \kappa(p) + (1-p)^{2d}\,, \tag{2.10}$$

where $\kappa(p)$ is the mean number density of clusters, see e.g. Chap. 4 in [13], and $(1-p)^{2d}$ the mean number density of isolated vertices.

(iii) The counterpart of Theorem 2.7 for the non-percolating phase, $p \in]0, p_c[$, was proved in [19]. There, N_{N} was shown to have a different kind of Lifshits asymptotics with a Lifshits exponent 1/2 at the lower spectral edge, see also Section 1, and the same is true for N_{D} at the upper spectral edge. This type of Lifshits behaviour is caused by large (isolated) linear clusters, which explains why the spatial dimension does not influence the Lifshits exponent. This behaviour is also present for $p \in]p_c, 1[$, but hidden under the more dominant van Hove asymptotics caused by the percolating cluster.

(iv) At the critical point $p = p_c$, the behaviour of N_{N} at the lower spectral edge, respectively that of N_{D} at the upper spectral edge, is an open problem.

3 Proof of Theorem 2.5

In this section we prove the Lifshits-tail behaviour of Theorem 2.5. Thanks to the symmetries (2.3), it suffices to consider the lower spectral edge only.

Lemma 3.1. *For every $d \in \mathbb{N}$ and every $p \in]0,1[$ there exist constants ε_{D}, α_u, $\alpha_l \in]0,\infty[$ such that*

$$\exp\{-\alpha_l E^{-d/2}\} \leqslant N_{\mathrm{D}}(E) \leqslant N_{\widetilde{\mathrm{D}}}(E) \leqslant \exp\{-\alpha_u E^{-d/2}\} \tag{3.1}$$

holds for all $E \in]0, \varepsilon_{\mathrm{D}}[$.

Proof. The left inequality in (3.1), i.e. the lower bound on N_{D}, was proved in Lemma 2.9 in [19]. The middle one simply reflects the operator inequality $\Delta_{\widetilde{\mathrm{D}}}^{(\omega)} \leqslant \Delta_{\mathrm{D}}^{(\omega)}$, which is valid for all $\omega \in \Omega$. So it remains to prove the upper bound on $N_{\widetilde{\mathrm{D}}}$.

We follow the strategy of the proof in [24], see also Chap. 2.1 in [25]. To do so, we have to fix some notation, first. Given a bounded cube $\Lambda \subset \mathbb{Z}^d$ and $x \in \Lambda$, we introduce the boundary degree

$$b_{\partial\Lambda}(x) := \Big|\Big\{\{x,y\} \in \mathbb{E}^d : y \notin \Lambda\Big\}\Big| \tag{3.2}$$

as the cardinality of the set of edges in the regular lattice $\mathbb{L}^d$ that connect x with $\mathbb{Z}^d \setminus \Lambda$. The restriction $\mathscr{G}_\Lambda^{(\omega)} := (\Lambda, \mathscr{E}_\Lambda^{(\omega)})$ with $\mathscr{E}_\Lambda^{(\omega)} := \Big\{\{x,y\} \in \mathscr{E}^{(\omega)} : x, y \in \Lambda\Big\}$ of any realisation $\mathscr{G}^{(\omega)}$ of a bond-percolation graph to Λ is obtained by keeping only vertices and edges within Λ, and $d_{\mathscr{G}_\Lambda^{(\omega)}}(x) \leqslant 2d - b_{\partial\Lambda}(x)$ stands for the associated vertex degree of $x \in \Lambda$. In particular, $\mathbb{E}_\Lambda^d := \Big\{\{x,y\} \in \mathbb{E}^d : x, y \in \Lambda\Big\}$ is the edge set of the fully connected cube $\mathbb{L}_\Lambda^d := (\Lambda, \mathbb{E}_\Lambda^d)$, that is the restriction of the regular lattice $\mathbb{L}^d$ to Λ. Finally, let $\ell^2(\Lambda)$ be the Hilbert space of complex-valued (finite) sequences indexed by Λ, and, given any subgraph $\mathfrak{G} := (\Lambda, \mathfrak{E})$ of $\mathbb{L}_\Lambda^d$, we introduce the operator $\mathfrak{H}_\mathfrak{G} : \ell^2(\Lambda) \to \ell^2(\Lambda)$, $\varphi \mapsto \mathfrak{H}_\mathfrak{G}\varphi$, where

$$
\begin{aligned}
\mathfrak{H}_\mathfrak{G}\varphi(x) &:= - \sum_{y\in\Lambda:\{x,y\}\in\mathfrak{E}} \varphi(y) \; + \Big(2d - b_{\partial\Lambda}(x)\Big)\varphi(x) \\
&= \sum_{y\in\Lambda:\{x,y\}\in\mathfrak{E}} \Big(\varphi(x) - \varphi(y)\Big) \; + \Big(2d - b_{\partial\Lambda}(x) - d_\mathfrak{G}(x)\Big)\varphi(x) \qquad (3.3)
\end{aligned}
$$

for all $\varphi \in \ell^2(\Lambda)$ and all $x \in \Lambda$. Now, we define the restriction of the Pseudo-Dirichlet Laplacian $\Delta_{\widetilde{\mathrm{D}}}$ to the cube Λ with Neumann conditions along the boundary $\partial\Lambda$ of Λ as the random bounded self-adjoint operator H_Λ with realisations $H_\Lambda^{(\omega)} := \mathfrak{H}_{\mathscr{G}_\Lambda^{(\omega)}}$ for all $\omega \in \Omega$.

Next we claim that

$$
N_{\widetilde{\mathrm{D}}}(E) = \inf_{\Lambda\subset\mathbb{Z}^d} \left[\frac{1}{|\Lambda|} \int_\Omega \mathbb{P}(\mathrm{d}\omega')\ \operatorname{trace}_{\ell^2(\Lambda)} \Theta(E - H_\Lambda^{(\omega')})\right] \qquad (3.4)
$$

holds for all $E \in \mathbb{R}$. This is so, because (i) the operator $H_\Lambda^{(\omega)}$ differs from the finite-volume restriction $\Delta_{\widetilde{\mathrm{D}},\Lambda}^{(\omega)}$ in Remark 2.4 (iii) by a perturbation whose rank is at most of the order of $|\partial\Lambda|$, the surface area of the cube Λ. Hence, (2.4) remains true for $\mathrm{X} = \widetilde{\mathrm{D}}$ and with $\Delta_{\widetilde{\mathrm{D}},\Lambda}^{(\omega)}$ being replaced by $H_\Lambda^{(\omega)}$ on its right-hand side,

$$
N_{\widetilde{\mathrm{D}}}(E) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \left[\frac{1}{|\Lambda|}\ \operatorname{trace}_{\ell^2(\Lambda)} \Theta(E - H_\Lambda^{(\omega)})\right] . \qquad (3.5)
$$

(ii) On the other hand, $H_\Lambda^{(\omega)}$ is designed in such a way that $H_{\Lambda_1}^{(\omega)} \oplus H_{\Lambda_2}^{(\omega)} \leqslant H_{\Lambda_1\cup\Lambda_2}^{(\omega)}$ holds on $\ell^2(\Lambda_1\cup\Lambda_2)$ for all bounded cubes $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ with $\Lambda_1\cap\Lambda_2 = \varnothing$ and for all $\omega \in \Omega$. Hence, $\Theta(E - H_\Lambda^{(\omega)})$ gives rise to a subergodic process and we conclude from the Ackoglu–Krengel subergodic theorem that the right-hand side of (3.5) equals the right-hand side of (3.4) – again for all continuity points of the limit and uniformly for ω in a set of probability one. (iii) From this we have (3.4) for all continuity points of both sides. But since both sides of (3.4) are right-continuous functions of E, equality holds for all $E \in \mathbb{R}$, and the derivation is complete.

From (3.4) we infer the upper bound

$$N_{\widetilde{\mathrm{D}}}(E) \leqslant \inf_{\Lambda \subset \mathbb{Z}^d} \mathbb{P}[E_\Lambda \leqslant E] \,, \tag{3.6}$$

where the non-negative random variable E_Λ stands for the smallest eigenvalue of the random operator H_Λ.

The aim is to obtain a simple large-deviation estimate for the probability in (3.6). This will be achieved with the help of analytic perturbation theory along the lines of [24], see also Sec. 4.1.10 in [25]. We write $H_{0,\Lambda} := \mathfrak{H}_{\mathbb{L}^d_\Lambda}$ for the Neumann Laplacian of the fully connected cube $\mathbb{L}^d_\Lambda$ and $W_\Lambda := H_\Lambda - H_{0,\Lambda}$. Given $t \in [0,1]$, we introduce

$$H_\Lambda(t) := H_{0,\Lambda} + tW_\Lambda \tag{3.7}$$

so that $H_\Lambda(1) = H_\Lambda$. We want to construct an upper bound for the probability that E_Λ is small. Denoting the bottom eigenvalue of $H_\Lambda(t)$ by $E_\Lambda(t)$, we use the following ideas.

(a) The function $[0,1] \ni t \mapsto E_\Lambda(t)$ is non-decreasing, $E_\Lambda(0) = 0$ and $E_\Lambda(1) = E_\Lambda$.

(b) This function can be linearised, if its argument is small enough. More precisely, there exist constants $\tau, \beta \in]0,\infty[$, which depend only on the spatial dimension d, such that

$$|E_\Lambda(t) - tE'_\Lambda(0)| \leqslant \beta t^2 |\Lambda|^{2/d} \tag{3.8}$$

for all $t \in [0, \tau|\Lambda|^{-2/d}]$. Here, we have set $E'_\Lambda(0) := \frac{\mathrm{d}}{\mathrm{d}t} E_\Lambda(t)\Big|_{t=0}$.

(c) The slope $E'_\Lambda(0)$ obeys a large-deviation estimate. Given any $\alpha \in]0, 1-p[$, there exists a constant $\gamma \in]0,\infty[$, which depends on p and d, such that

$$\mathbb{P}[E'_\Lambda(0) \leqslant \alpha] \leqslant \mathrm{e}^{-\gamma|\Lambda|} \,. \tag{3.9}$$

We will prove (a) with a Perron–Frobenius argument in Lemma 3.2 and discuss observations (b) and (c) below. For the time being, let us go on to estimate the probability that E_Λ is small.

Suppose $E_\Lambda(t) \leqslant E$. Then we conclude from (a), the triangle inequality and (b) that

$$E'_\Lambda(0) \leqslant \frac{E_\Lambda(t)}{t} + \left|\frac{E_\Lambda(t)}{t} - E'_\Lambda(0)\right| \leqslant \frac{E}{t} + \beta t |\Lambda|^{2/d} \,, \tag{3.10}$$

provided t is small enough. So we need to adjust $t \equiv t_E$ and $\Lambda \equiv \Lambda_E$ such that $t_E \leqslant \tau|\Lambda_E|^{-2/d}$. Moreover, we aim to achieve that the right-hand side of (3.10) is bounded from above by some $\alpha < 1-p$. This is accomplished in the

following way. Without restriction we can assume that, in addition, $\alpha < 2\beta\tau$. Then we set $t_E := \alpha/(2\beta|\Lambda_E|^{2/d})$ and choose the size of the cube such that

$$\frac{\alpha}{2(\beta E)^{1/2}} - 1 \leqslant |\Lambda_E|^{1/d} \leqslant \frac{\alpha}{2(\beta E)^{1/2}} \,. \tag{3.11}$$

For this to make sense, the right-hand side of (3.11) has to exceed 2. So, we restrict ourselves to low energies, say $E \in]0, \varepsilon_{\mathrm{D}}[$, and summarise this argument as

$$E_{\Lambda_E}(t_E) \leqslant E \qquad \text{implies} \qquad E'_{\Lambda_E}(0) \leqslant \alpha < 1 - p \,. \tag{3.12}$$

Note that ε_{D} depends only on p and d.

Altogether, we infer from Eq. (3.6), observation (a), implication (3.12) and observation (c) that

$$N_{\widetilde{\mathrm{D}}}(E) \leqslant \mathbb{P}[E_{\Lambda_E}(t_E) \leqslant E] \leqslant \mathbb{P}[E'_{\Lambda_E}(0) \leqslant \alpha] \leqslant \mathrm{e}^{-\gamma|\Lambda_E|} \leqslant \mathrm{e}^{-\alpha_u E^{-d/2}} \,, \tag{3.13}$$

where $\alpha_u \in]0, \infty[$ is a constant that depends only on p and d.

Next, we verify observations (b) and (c) above. Observation (b) relies on a deterministic result from analytic perturbation theory. To this end we consider the operator family $H(z) := H_0 + zH_1$ for $z \in \mathbb{C}$. Here, $H_0 := H_{0,\Lambda}$ is the Neumann Laplacian of $\mathbb{L}^d_\Lambda$ and $H_1 := W^{(\omega)}_\Lambda$ the perturbation with $\omega \in \Omega$ arbitrary, but fixed. The bottom eigenvalue 0 of H_0 is an isolated simple eigenvalue. Its isolation distance $\vartheta := \mathrm{dist}\big(0, \mathrm{spec}(H_0 \setminus \{0\})\big)$ is determined by the magnitude of the smallest non-zero eigenvalue of H_0. This distance obeys the estimate $\vartheta \geqslant c|\Lambda|^{-2/d}$ for some constant $c \in]0, \infty[$, which follows from reducing the eigenvalue problem for H_0 to that of a linear chain by separation of variables and applying a Cheeger-type inequality, see e.g. (2.6) in [19]. Moreover, we have the uniform bound $\|W^{(\omega)}_\Lambda\| \leqslant 2d$ for the operator norm of the perturbation so that $H(z)$ has one isolated eigenvalue $E(z)$ in the complex disc $B_{\vartheta/2}(0)$ provided $z < \vartheta/(4d)$. We refer to [17], Chap. II, §1, Secs. 1, 2 and Chap. VII, §3, Secs. 1, 2 and 4 for a detailed exposition of the general method. Elementary function theory then gives an estimate for the second derivative of $E(z)$, and Taylor's theorem yields (b). Details of the argument, geared towards our application here, can also be found in Sec. 4.1.10 in [25].

Concerning observation (c), we refer again to analytic perturbation theory. The Feynman–Hellmann formula yields

$$E'_\Lambda(0) = \langle \varphi_0, W_\Lambda \varphi_0 \rangle \,, \tag{3.14}$$

where $\varphi_0 := |\Lambda|^{-1/2}$, the normalised vector in $\ell^2(\Lambda)$ with equal components, is the ground state of the unperturbed operator $H_{0,\Lambda}$. Therefore, recalling

$W_\Lambda = H_\Lambda - H_{0,\Lambda} = \mathfrak{H}_{\mathscr{G}_\Lambda} - \mathfrak{H}_{\mathbb{L}^d_\Lambda}$ and the definition in (3.3), we have

$$\begin{aligned} E^{(\omega)}_\Lambda{}'(0) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda: \\ \{x,y\}\in\mathbb{E}^d_\Lambda\setminus\mathscr{E}^{(\omega)}_\Lambda}} 1 \; &= \; \frac{2}{|\Lambda|} \sum_{\{x,y\}\in\mathbb{E}^d_\Lambda} (1-\omega_{\{x,y\}}) \\ &\geqslant \frac{1}{|\mathbb{E}^d_\Lambda|} \sum_{\{x,y\}\in\mathbb{E}^d_\Lambda} (1-\omega_{\{x,y\}}) \end{aligned} \tag{3.15}$$

for all $\omega \in \Omega$. We recall that the $\omega_{\{x,y\}}$'s, which indicate the presence of an edge in the bond-percolation graph, are i.i.d. Bernoulli distributed with mean p. Hence, (3.9) follows from standard large-deviation estimates, see e.g. inequality (27.4) in [16] or Thm. 1.4 in [26]. $\square$

So far we have deferred the proof of observation (a) in the above demonstration. This is a deterministic result which we address now in

Lemma 3.2. *Let $\Lambda \subset \mathbb{Z}^d$ be a bounded cube, let $\mathfrak{G} = (\Lambda, \mathfrak{E})$ be a subgraph of the fully connected cube $\mathbb{L}^d_\Lambda$ and let $\mathfrak{H}_\mathfrak{G}$ be the finite-volume Laplacian* (3.3) *on $\ell^2(\Lambda)$. For $t \in \mathbb{R}$ let $\mathfrak{e}(t)$ be the smallest eigenvalue of*

$$\mathfrak{h}(t) := \mathfrak{H}_{\mathbb{L}^d_\Lambda} + t\,\mathfrak{W}\,, \tag{3.16}$$

where $\mathfrak{W} := \mathfrak{H}_\mathfrak{G} - \mathfrak{H}_{\mathbb{L}^d_\Lambda}$. Then the function $[0,1] \ni t \mapsto \mathfrak{e}(t)$ is non-decreasing.

Proof. We observe from the definition of $\mathfrak{W}$ and (3.3) that

$$\mathfrak{W}\,\varphi(x) = \sum_{y\in\Lambda:\{x,y\}\in\mathbb{E}^d_\Lambda\setminus\mathfrak{E}} \varphi(y) \tag{3.17}$$

for all $\varphi \in \ell^2(\Lambda)$ and all $x \in \Lambda$. Given $t \in [0,1]$ let us rewrite $\mathfrak{h}(t) = \mathfrak{H}_\mathfrak{G} - (1-t)\,\mathfrak{W} =: 2d1\!\!1 - \mathfrak{a}(t)$. In particular, $\mathfrak{a}(1) = 2d1\!\!1 - \mathfrak{H}_\mathfrak{G}$ acts as

$$\mathfrak{a}(1)\,\varphi(x) = \sum_{y\in\Lambda:\{x,y\}\in\mathfrak{E}} \varphi(y) \;+ b_{\partial\Lambda}(x)\varphi(x) \tag{3.18}$$

for all $\varphi \in \ell^2(\Lambda)$ and all $x \in \Lambda$. Equations (3.17) and (3.18) show that the self-adjoint linear operator

$$\mathfrak{a}(t) = \mathfrak{a}(1) + (1-t)\,\mathfrak{W}\,, \tag{3.19}$$

which lives on the finite-dimensional Hilbert space $\ell^2(\Lambda)$, has only non-negative matrix elements $\langle\delta_x, \mathfrak{a}(t)\delta_y\rangle$ for all $x, y \in \Lambda$. Together with the min-max principle, this implies that one can choose the eigenvector(s) corresponding to the *largest* eigenvalue of $\mathfrak{a}(t)$ in such a way that all their components

in the basis $\{\delta_x\}_{x\in\Lambda}$ are non-negative. Hence, the same is true for the eigenvector(s) corresponding to the *smallest* eigenvalue of $\mathfrak{h}(t)$. Thus, another application of the min-max principle yields

$$\begin{aligned}\mathfrak{e}(t_2) &= \inf_{\substack{0\neq\varphi\in\ell^2(\Lambda)\\ \varphi(x)\geqslant 0\ \ \forall x\in\Lambda}} \frac{\langle\varphi,\mathfrak{h}(t_2)\varphi\rangle}{\langle\varphi,\varphi\rangle} = \inf_{\substack{0\neq\varphi\in\ell^2(\Lambda)\\ \varphi(x)\geqslant 0\ \ \forall x\in\Lambda}} \frac{\langle\varphi,\mathfrak{h}(t_1)\varphi\rangle + (t_2-t_1)\langle\varphi,\mathfrak{W}\,\varphi\rangle}{\langle\varphi,\varphi\rangle}\\ &\geqslant \mathfrak{e}(t_1)\end{aligned} \tag{3.20}$$

for all $0\leqslant t_1\leqslant t_2\leqslant 1$, because the scalar product involving $\mathfrak{W}$ is non-negative by (3.17). $\square$

4 Proof of Theorem 2.7

In this section we prove the van Hove asymptotics of Theorem 2.7. Again, it suffices to consider the lower spectral edge, because of the symmetries (2.3). That asymptotics follows from

Lemma 4.1. *Assume $d\in\mathbb{N}\setminus\{1\}$ and $p\in]p_c,1[$. Then there exist constants ε_{N}, C_u, $C_l\in]0,\infty[$ such that*

$$C_l E^{d/2}\leqslant N_{\mathrm{N}}(E)-N_{\mathrm{N}}(0)\leqslant C_u E^{d/2} \tag{4.1}$$

holds for all $E\in]0,\varepsilon_{\mathrm{N}}[$.

To prove Lemma 4.1 we separate the contribution of the percolating cluster to N_{N} from that of the finite clusters.

Definition 4.2. Let Ω_∞ denote the event that the origin belongs to the percolating cluster and, for $E\in\mathbb{R}$, define

$$N_{\mathrm{N},\infty}(E) := \int_{\Omega_\infty}\mathbb{P}(\mathrm{d}\omega)\ \langle\delta_0,\Theta\Big(E-\Delta_{\mathrm{N}}^{(\omega)}\Big)\delta_0\rangle\,, \tag{4.2}$$

which is the contribution of the percolating cluster to the integrated density of states of the Neumann Laplacian. We write $\widetilde{N}_{\mathrm{N},\infty}(t) := \int_0^\infty \mathrm{d}\,N_{\mathrm{N},\infty}(E)\ \mathrm{e}^{-Et}$ for its Laplace transform, where $t\in[0,\infty[$.

As is well known, the Laplace transform of (4.2) can be related to the mean return probability of a continuous-time, simple random walk $\{Z_t\}_{t\in[0,\infty[}$ on the percolating cluster. More precisely, this random walk is the Markov process on $\mathbb{Z}^d$ defined by the following set of rules: Suppose the process is at $x\in\mathbb{Z}^d$. Then, after having waited there for an exponential time of parameter one, one of the $2d$ neighbours of x in $\mathbb{Z}^d$, say y, is chosen at random with probability $1/(2d)$. If $\omega_{\{x,y\}}=1$, then the process jumps immediately to y,

otherwise there will be no move. The procedure then starts afresh. Assuming that $Z_0 = x_0 \in \mathbb{Z}^d$ is the starting point of the process, we denote its law by $\mathscr{P}^{(\omega)}_{x_0}$. The process Z_t is generated by the Neumann Laplacian in the sense that the transition probability for going from x to y within time t is given by $\mathscr{P}^{(\omega)}_{x_0}(Z_{s+t} = y \,|\, Z_s = x) = \mathscr{P}^{(\omega)}_{x}(Z_t = y) = \langle \delta_y, \mathrm{e}^{-t\Delta^{(\omega)}_{\mathrm{N}}/(2d)}\delta_x\rangle$ for all $s \in [0,\infty[$ and all x_0 in the same connected component as x and y. From this it follows that

$$\widetilde{N}_{\mathrm{N},\infty}(t) = \int_{\Omega_\infty} \mathbb{P}(\mathrm{d}\omega)\ \mathscr{P}^{(\omega)}_0(Z_{2dt} = 0)\,. \tag{4.3}$$

Hence, $\big(\mathbb{P}(\Omega_\infty)\big)^{-1}\widetilde{N}_{\mathrm{N},\infty}(t)$ is the (conditional) mean return probability at time $2dt$ for the process on the percolating cluster for $p \in]p_c, 1]$.

Averaged transition probabilities of Z_t or related random walks have recently been studied in [21,2,14] with elaborate methods. We state a special case of the results as

Proposition 4.3. *Assume $d \in \mathbb{N}\setminus\{1\}$ and $p \in]p_c, 1[$. Then there exist constants $c_l, c_u \in]0,\infty[$ and $t_0 \in]1,\infty[$, all of which depend only on p and d, such that*

$$c_l t^{-d/2} \leqslant \widetilde{N}_{\mathrm{N},\infty}(t) \leqslant c_u t^{-d/2} \tag{4.4}$$

holds for all $t \in [t_0, \infty[$.

Remark 4.4. In view of (4.3), the lower bound in Proposition 4.3 can be found as Eq. (30) in Appendix D in [21]. That paper also contains a "quenched" upper bound, i.e. an upper bound for $\mathscr{P}^{(\omega)}_x(Z_t = y)$, which is valid for $\mathbb{P}$-almost every $\omega \in \Omega_\infty$. Unfortunately, it is not clear how to take the probabilistic expectation thereof, that is, to get an "annealed" upper bound. On the other hand, the authors of [14] prove an annealed upper bound in their Thm. 8.1. But this bound includes an additional logarithmic factor. The strongest results, both annealed and quenched, are those of Barlow [2], and (4.4) is a special case thereof. However, his results apply to a random walk which is generated by $D^{-1}\Delta_{\mathrm{N}}$ instead of Δ_{N}. Hence, some additional comments are needed to adapt his results, and we address this issue now.

Proof of Proposition 4.3. According to the preceding remark the proposition is established, if we show that Barlow's quenched upper bound for the return probability, i.e. the special case $x = y$ of the upper bound in Thm. 2 in [2], applies also to the random walk generated by Δ_{N}. Eventually, the upper bound in Thm. 2 in [2] is reduced to Prop. 3.1 in [2] via Thm. 1, Prop. 6.1, Thm. 5.7 and Thm. 3.8 – the latter being nothing but the off-diagonal generalisation of Prop. 3.1, so we do not need it here. The reduction does not make use of any specific properties of the random walk's generator. Hence, all that remains to check is Prop. 3.1. in [2]. It turns out that some of the constants in the proof of Prop. 3.1 must be modified for our purpose, but this does not have any consequences. In addition, the proof of Prop. 3.1 also requires estimates (1.7)

and (1.8) of Lemma 1.1 in [2]. Estimate (1.8) follows from estimate (1.5), as is argued in the proof of Lemma 1.1(b) in [2]. This argument applies in our situation, too. So, in the end, we have to verify the validity of (1.7) and (1.5) in [2] for the random walk generated by Δ_{N}.

Estimate (1.5) is seen to hold as a special case of Cor. 11 in [12]. The upper bound in estimate (1.7) can be inferred, for example, from Thm. II.5 in [8], which is a general result for ultracontractive Markov semigroups. (Actually, we refer to the first of the two theorems with the same number II.5 in [8].) To verify statement (ii) of that theorem, one may use the application following it together with the reasoning in Cor. V.2 and its proof with the choice $\psi(x) = x$. This choice corresponds to a weak isoperimetric inequality which merely reflects that the percolating cluster contains an infinitely long path. From this point of view, the weak $t^{-1/2}$-decay of the upper bound in estimate (1.7) does not come as a surprise. Finally, the lower bound in estimate (1.7) arises solely from the uniform growth condition $|B(x,R)| \leqslant \mathrm{const.}R^d$ for the volume of a ball around $x \in \mathbb{Z}^d$ with radius R. Such type of results are well known for heat kernels on manifolds and also for discrete-time random walks [20,9,10]. To employ them here, we decompose

$$\mathscr{P}_x(Z_t = x) = \sum_{n=0}^{\infty} \langle \delta_x, K^n \delta_x \rangle \; \mathscr{P}_x \begin{pmatrix} \text{there are } n \text{ attempted} \\ \text{jumps up to time } t \end{pmatrix}, \tag{4.5}$$

using the stochastic independence of all building blocks of Z_t. Here, the contraction $K := 1\!\!1 - \Delta_{\mathrm{N}}/(2d)$ is the transition matrix of a discrete-time random walk on $\mathbb{Z}^d$, which controls the directions of the jumps of Z_t. The number of attempted jumps up to time t is governed by a Poisson distribution with mean t. Using this, the lower bound in (1.7) follows from Thm. 3(ii) in [20] applied to K. $\square$

In order to apply Proposition 4.3 in the proof of Lemma 4.1, we will use two elementary Tauberian inequalities.

Lemma 4.5. *Let μ be a positive Borel measure on $\mathbb{R}^+ = [0,\infty[$. Suppose there are constants $t_0, \delta, c_l, c_u \in]0,\infty[$ such that the Laplace transform $\widetilde{\mu}(t) := \int_{\mathbb{R}^+} \mu(\mathrm{d}E)\, \mathrm{e}^{-Et}$ exists for all $t \in [t_0, \infty[$ and obeys*

$$c_l t^{-\delta} \leqslant \widetilde{\mu}(t) \leqslant c_u t^{-\delta}\,. \tag{4.6}$$

Then there exist constants $C_l, C_u \in]0,\infty[$ such that

$$C_l E^{\delta} \leqslant \mu\Big([0,E]\Big) \leqslant C_u E^{\delta} \tag{4.7}$$

holds for all $E \in]0, t_0^{-1}]$.

Proof. We express $\mu([0,E]) = \int_{\mathbb{R}^+} \mu(\mathrm{d}\lambda)\,\Theta(1-\lambda/E)$ in terms of the right-continuous Heaviside unit-step function and observe the elementary inequality $\mathrm{e}^{-\tau x} - \mathrm{e}^{-(\tau-1)}\mathrm{e}^{-x} \leqslant \Theta(1-x) \leqslant \mathrm{e}^{1-x}$, which is valid for all $x \in \mathbb{R}^+$ and all $\tau \in [1,\infty[$. The upper bound in (4.7) is obvious now. For the lower bound, one has to choose τ large enough such that the constant, which arises from the application of both estimates in (4.6), is strictly positive. □

Proof of Lemma 4.1. We set $N_{\mathrm{N,fin}} := N_{\mathrm{N}} - N_{\mathrm{N},\infty}$ for the contribution of the finite clusters to the integrated density of states and observe

$$N_{\mathrm{N}}(E) - N_{\mathrm{N}}(0) = [N_{\mathrm{N,fin}}(E) - N_{\mathrm{N,fin}}(0)] + N_{\mathrm{N},\infty}(E) \tag{4.8}$$

for all $E \in \mathbb{R}$.

Proposition 4.3 and Lemma 4.5 establish the desired van Hove bounds for $N_{\mathrm{N},\infty}$. Therefore, it remains to show that the finite clusters do not spoil this behaviour. Since $N_{\mathrm{N,fin}}(E) - N_{\mathrm{N,fin}}(0) \geqslant 0$ for all $E \in [0,\infty[$, only an appropriate upper bound is required for the contribution of the finite clusters. We shall show in Eq. (4.10) below that $N_{\mathrm{N,fin}}(E) - N_{\mathrm{N,fin}}(0)$ obeys even a Lifshits-type upper bound, which will then complete the proof.

The Lifshits behaviour for the contribution of the finite clusters in the percolating phase arises from the cluster-size distribution – in the same way as it was shown to arise in the non-percolating phase in [19]. Indeed, we have

$$N_{\mathrm{N,fin}}(E) - N_{\mathrm{N,fin}}(0) \leqslant \mathbb{P}\Big\{\omega \in \Omega_{\mathrm{fin}} : |\mathscr{C}_0^{(\omega)}| \geqslant (dE)^{-1/2}\Big\} \tag{4.9}$$

for all $E \in]0,\infty[$. Here $\Omega_{\mathrm{fin}} := \Omega \setminus \Omega_\infty$ is the event that the origin belongs to a finite cluster, say $\mathscr{C}_0^{(\omega)}$, and $|\mathscr{C}_0^{(\omega)}|$ denotes the number of its vertices. Inequality (4.9) follows from repeating the steps that lead to the first inequality in Eq. (2.24) in [19] with $N_{\mathrm{N,fin}}$ instead of N_{N}. For $p > p_c$ the cluster-size distribution on the right-hand side of (4.9) decays sub-exponentially according to Thm. 8.6.5 in [13] so that we obtain

$$N_{\mathrm{N,fin}}(E) - N_{\mathrm{N,fin}}(0) \leqslant c_1 \exp\{-\xi E^{-(d-1)/2d}\} \tag{4.10}$$

for all $E \in]0,\infty[$ with some constants $c_1, \xi \in]0,\infty[$, which depend only on p and d. This completes the proof. □

References

[1] P. Antal, Enlargement of obstacles for the simple random walk, Ann. Probab. 23 (1995) 1061–1101.

[2] M. T. Barlow, Random walks on supercritical percolation clusters, Ann. Probab. 32 (2004) 3024–3084.

[3] M. Barlow, T. Coulhon, A. Grigor'yan, Manifolds and graphs with slow heat kernel decay, Invent. Math. 144 (2001) 609–649.
[4] M. Biskup, W. König, Long-time tails in the parabolic Anderson model with bounded potential, Ann. Probab. 29 (2001) 636–682.
[5] Fan R. K. Chung, Spectral graph theory, American Mathematical Society, Providence, RI, 1997.
[6] Fan Chung, A. Grigor'yan, Shing-Tung Yau, Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs, Commun. Anal. Geom. 8 (2000) 969–1026.
[7] Y. Colin de Verdière, Spectres de graphes, Société Mathématique de France, Paris, 1998 [in French].
[8] T. Coulhon, Ultracontractivity and Nash type inequalities, J. Funct. Anal. 141 (1996) 510–539.
[9] T. Coulhon, A. Grigor'yan, On-diagonal lower bounds for heat kernels and Markov chains, Duke Math. J. 89 (1997) 133–199.
[10] T. Coulhon, A. Grigor'yan, Random walks on graphs with regular volume growth, Geom. Funct. Anal. 8 (1998) 656–701.
[11] D. Cvetković, M. Domb, H. Sachs, Spectra of graphs: Theory and applications, Johann Ambrosius Barth, Heidelberg, 1995.
[12] E. B. Davies, Large deviations for heat kernels on graphs, J. London Math. Soc. (2) 47 (1993) 62–72.
[13] G. Grimmett, Percolation, 2nd ed., Springer, Berlin, 1999.
[14] D. Heicklen, C. Hoffman, Return probabilities of a simple random walk on percolation clusters, Electronic J. Probab. 10 (2005) 250–302.
[15] M. Kac, Can one hear the shape of a drum?, Am. Math. Mon. 73 (4-II) (1966) 1–23.
[16] O. Kallenberg, Foundations of modern probability, 2nd ed., Springer, New York, 2001.
[17] T. Kato, Perturbation theory for linear operators, 2nd ed., Springer, Berlin, 1976.
[18] O. Khorunzhy, W. Kirsch, P. Müller, Lifshits tails for spectra of Erdős–Rényi random graphs, Ann. Appl. Probab. 16 (2006) 295–309.
[19] W. Kirsch, P. Müller, Spectral properties of the Laplacian on bond-percolation graphs, Math. Z. 252 (2006) 899–916.
[20] F. Lust-Piquard, Lower bounds on $\|K^n\|_{1\to\infty}$ for some contractions K of $L^2(\mu)$, with applications to Markov operators, Math. Ann. 303 (1995) 699–712.
[21] P. Mathieu, E. Remy, Isoperimetry and heat kernel decay on percolation clusters, Ann. Probab. 32 (2004) 100–128.
[22] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[23] B. Mohar, The Laplacian spectrum of graphs, graph theory, combinatorics, and applications, Wiley, New York, 1991.
[24] P. Stollmann, Lifshitz asymptotics via linear coupling of disorder, Math. Phys. Anal. Geom. 2 (1999) 279–289.
[25] P. Stollmann, Caught by disorder: bound states in random media, Birkhäuser, Boston, 2001.
[26] M. Talagrand, New concentration inequalities in product spaces, Invent. Math. 126 (1996) 505–563.